

Transition Property For Cube-Free Words

Elena A. Petrova¹

Ural Federal University, Ekaterinburg, Russia
elena.petrova@urfu.ru

Arseny M. Shur²

Ural Federal University, Ekaterinburg, Russia
arseny.shur@urfu.ru

Abstract

We study cube-free words over arbitrary non-unary finite alphabets and prove the following structural property: for every pair (u, v) of d -ary cube-free words, if u can be infinitely extended to the right and v can be infinitely extended to the left respecting the cube-freeness property, then there exists a “transition” word w over the same alphabet such that uwv is cube free. The crucial case is the case of the binary alphabet, analyzed in the central part of the paper.

The obtained “transition property”, together with the developed technique, allowed us to solve cube-free versions of three old open problems by Restivo and Salemi. Besides, it has some further implications for combinatorics on words; e.g., it implies the existence of infinite cube-free words of very big subword (factor) complexity.

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1 Introduction

The concept of power-freeness is in the center of combinatorics on words. This concept expresses the restriction on repeated blocks (factors) inside a word: an α -power-free word contains no block which consecutively occurs in it α or more times. For example, the block **an** in the word **banana** is considered as having $5/2$ consecutive occurrences; thus the word **banana** is 3-power-free (cube-free) but not $(5/2)$ -power-free or 2-power-free (square-free); the block **mag** in the word **magma** occurs consecutively $5/3$ times, and so **magma** is square-free. Power-free words and languages are studied in lots of papers starting with the seminal works by Thue [19, 20], who proved, in particular, the infiniteness of the sets of binary cube-free words and ternary square-free words. However, many phenomena related to power-freeness are still not understood.

One group of problems about power-free words concerns their structure and extendability. In 1985, Restivo and Salemi presented [15] a list of five problems, originally considered only for ternary square-free words and binary overlap-free words, but equally important for every power-free language. Suppose that a finite alphabet Σ is fixed and we study α -power-free words over Σ . Here are the problems.

Problem 1. Given an α -power-free word u , decide whether there are infinitely many α -power-free words having (a) the prefix u ; or (b) the suffix u ; or (c) the form vuw , where v and w have equal length. (Such words u are called, respectively, right extendable, left extendable, and two-sided extendable.)

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Problem 2. Given an α -power-free word u , construct explicitly an α -power-free infinite word having u as prefix, provided that u is right extendable.

Problem 3. Given an integer $k \geq 0$, does there exist an α -power-free word u with the properties (i) there exists a word v of length k such that uv is α -power free and (ii) for every word v' of bigger length, uv' is not α -power free.

Problem 4. Given two α -power-free words u and v , decide whether there is a “transition” from u to v (i.e., does there exist a word w such that uwv is α -power free).

Problem 5. Given two α -power-free words u and v , find explicitly a transition word w , if it exists.

These natural problems appear to be rather hard. Only for Problem 1a,b there is a sort of a general solution: a backtracking decision procedure exists for all k -power-free languages, where $k \geq 2$ is an integer [3,4]. In a number of cases, the parameters of backtracking were found by computer search, so it is not clear whether this technique can be extended for α -power-free words with rational α . The decision procedure also gives no clue to Problem 2.

There is a particular case of binary overlap-free words, for which all problems are solved in [1,15] (more efficient solutions were given in [2]). These words have a regular structure deeply related to the famous Thue-Morse word, and it seems that all natural algorithmic problems for them are solved. For example, the asymptotic order of growth for the binary overlap-free language is computed exactly [6,7], and even the word problem in the corresponding syntactic monoid has a linear-time solution [18]. Most of the results can be extended, with additional technicalities, to binary α -power-free words for any $\alpha \leq 7/3$, because the structure of these words is essentially the same as of overlap-free words (see, e.g., [8]). However, the situation changes completely if we go beyond the polynomial-size language of binary $(7/3)$ -power-free words. In the exponential-size α -power-free languages³ the diversity of words is much bigger, so it becomes harder to find a universal decision procedure. The only results on Problems 1-5 apart from those mentioned above are the positive answers to Problem 3 (including its two-sided analog) for the two classical test cases: for ternary square-free words [12] and for binary cube-free words [11].

In this paper, we study cube-free words over arbitrary alphabets. Still, the crucial case is the one of the binary alphabet; the central part of the paper is the proof of the following transition property of binary cube-free words.

► **Theorem 1.** *For every pair (u, v) of binary cube-free words such that u is right extendable and v is left extendable, there exists a binary word w such that uwv is cube free.*

After proving Theorem 1 in Section 3, we use it and its proof to derive further results. In Section 4 we prove the transition property for arbitrary alphabets (Theorem 9), while in Section 5 we use this property to solve Restivo-Salemi Problems 2, 4, and 5. Thus, all Restivo-Salemi problems for binary cube-free words are solved; this is the first fully solved case since the original publication of the problems. For cube-free words over bigger alphabets, only Problem 3 is not yet solved.

We finish the introduction with two remarks. First, the result of Theorems 1 and 9 was conjectured, in a slightly weaker form, for all infinite power-free languages [17, Conj 1]. This conjecture is related to the properties of finite automata recognizing some approximations of power-free languages and was supported by extensive numerical studies. The transition words

³ For $\alpha > 7/3$, the language of binary α -power-free words has exponential size [8]. *The exponential conjecture* says that for $k \geq 3$ all infinite power-free languages over k letters have exponential size. This conjecture is proved for $k \leq 10$ [9,10] and odd k up to 101 [21].

can be naturally interpreted as transitions in those automata and the transition property forces the automata to be strongly connected. Second, recently it was shown [16, Thm 39] that the transition property implies the existence of infinite α -power-free words of very big subword complexity. Namely, Theorems 1 and 9 imply that for every $d \geq 2$ there exists a d -ary cube-free infinite word which contains *all* two-sided extendable d -ary cube-free finite words as factors.

2 Preliminaries

2.1 Notation and Definitions

By default, we study words over finite alphabets Σ_d of cardinality $d \geq 2$, writing $\Sigma_d = \{a, b, c_1, \dots, c_{d-2}\}$ (mostly we work with $\Sigma_2 = \{a, b\}$). Standard notions of factor, prefix, and suffix are used. The set of all finite (nonempty finite, infinite) words over an alphabet Σ is denoted by Σ^* (resp., Σ^+ , Σ^∞). Elements of Σ^+ (Σ^∞) are treated as functions $w : \{1, \dots, n\} \rightarrow \Sigma$ (resp., $\mathbf{w} : \mathbb{N} \rightarrow \Sigma$). We write $[i..j]$ for the range $i, i+1, \dots, j$ of positive integers; the notation $w[i..j]$ stands for the factor of the word w occupying this range as well as for the particular occurrence of this factor in w at position i . Note that $w[i..i] = w[i]$ is just the i th letter of w . Let $w[i_1..j_1]$ and $w[i_2..j_2]$ be two factors of w . If the ranges $[i_1..j_1]$ and $[i_2..j_2]$ have a nonempty intersection, their intersection and union are also ranges; we refer to the factors of w , occupying these ranges, as the *intersection* and the *union* of $w[i_1..j_1]$ and $w[i_2..j_2]$. The word $\overleftarrow{w} = w[n] \dots w[1]$ is called the *reversal* of the word w of length n .

We write λ for the empty word and $|w|$ for the length of a word w (infinite words have length ∞). A word w has *period* $p < |w|$ if $w[1..|w|-p] = w[p+1..|w|]$; the prefix $w[1..p]$ of w is the *root* of this period of w . One of the most useful properties of periodic words is the following.

► **Lemma 2** (Fine, Wilf [5]). *If a word u has periods p and q and $|u| \geq p + q - \gcd(p, q)$ then u has period $\gcd(p, q)$.*

A *cube* is a nonempty word of the form uuu , also written as u^3 ; we refer to u as the root of this cube. A word is *cube-free* (*overlap-free*) if it has no cubes as factors (resp., no factors of the form $cwcwc$, where c is a letter). There exist binary overlap-free (and thus cube-free) infinite words [20].

Let Σ_d be fixed. A word $w \in \Sigma_d^*$ is called a *right context* of a cube-free word $u \in \Sigma_d^*$ if uw is cube free; we call u *right extendable* if it has an infinite right context (or, equivalently, infinitely many right contexts). Left contexts and left extendability are defined in a dual way.

The *Thue–Morse morphism* θ is defined over Σ_2^+ by the rules $\theta(a) = ab$, $\theta(b) = ba$. The fixed points of θ are the infinite *Thue–Morse word*

$$\mathbf{T} = abbabaabbaababbabaababbaabbabaab \dots$$

and its complement, obtained from \mathbf{T} by exchanging a 's and b 's. We refer to the factors of \mathbf{T} as *Thue–Morse factors*. The word \mathbf{T} , first introduced by Thue in [20] and rediscovered many times, possesses a huge number of nice properties; we need just a few. The Thue–Morse word is overlap free, *uniformly recurrent* (every Thue–Morse factor occurs in \mathbf{T} infinitely many times with a bounded gap), and *closed under reversals* (u is a Thue–Morse factor iff \overleftarrow{u} is).

2.2 Uniform words and markers

We call a word $w \in \Sigma_2^*$ *uniform* if $w = c\theta(u)d$ for some $c, d \in \{a, b, \lambda\}$, $u \in \Sigma_2^*$; a uniform word with $d = \lambda$ is *right aligned*. Similarly, a uniform infinite word has the form $c\theta(\mathbf{u})$ for

$c \in \{a, b, \lambda\}$, $u \in \Sigma_2^\infty$. Such “almost” θ -images play a crucial role in further considerations. Note that all factors and suffixes of \mathbf{T} are uniform. The following observation is well known.

► **Observation 1.** A word $u \in \Sigma_2$ is uniform iff all occurrences of factors of the form cc in u , where $c \in \Sigma_2$, are at positions of the same parity.

Thus the word is non-uniform iff it has the factors of the form cc occurring in positions of different parity. The following observation is straightforward.

► **Observation 2.** A cube-free word $u \in \Sigma_2$ is non-uniform iff it contains at least one of the factors $aabaa$, $ababaa$, $bbabb$, $bbababb$.

All right (resp., left) contexts of the word $ababa$ begin (resp., end) with a , so $ababa$ occurs in a cube-free word only as a prefix/suffix or inside the non-uniform factor $aababaa$ (same argument applies to $babab$). This allows us to view binary cube-free words as sequences of uniform factors separated by *markers* $aabaa$, $ababa$, $babab$, and $bbabb$, which break uniformity.

The importance of markers for the analysis of cube-free words is demonstrated by the following theorem, proved in Section 3.2.

► **Theorem 3.** *Every right-extendable cube-free word $u \in \Sigma_2$ has an infinite right context with finitely many markers.*

3 Proof of the Transition Property for Binary Words

The proof of Theorem 1 consists of two stages. In the first stage we show that its result is implied by Theorem 3. In the second stage we prove Theorem 3. All words in this section are over Σ_2 if the converse is not stated explicitly.

3.1 Reduction to Theorem 3

► **Lemma 4.** *Suppose that cube-free words u and v have right contexts which are Thue-Morse factors of length $2|u|$ and $2|v|$ respectively. Then there exists a word w such that $uw\overleftarrow{v}$ is cube free.*

Proof. Let u_1 and v_1 be the mentioned contexts of u and v respectively. Since \mathbf{T} is recurrent and closed under reversals, there exists a Thue-Morse factor $w = u_1 w_1 \overleftarrow{v_1}$ for some $w_1 \neq \lambda$. Assume to the contrary that $uw\overleftarrow{v}$ contains a cube of period p . Since w is overlap free, such a cube cannot intersect u and \overleftarrow{v} simultaneously by the length argument. W.l.o.g. assume that the cube intersects u . Then it must contain the whole u_1 . Hence p is a period of u_1 and thus $p \geq |u_1|/2 \geq |u|$ because u_1 is overlap free. Further, the overlap-freeness of w means that u contains at least the whole period of the cube, implying $p \leq |u|$. So we have $|u| = p$ and the cube is a prefix of $uw\overleftarrow{v}$. But in this case $|u_1| \geq 2p$ and the cube is contained in the cube-free word uu_1 , resulting in a contradiction. ◀

We say that a cube-free word u is *T-extendable* if it has a right context of the form $w\mathbf{T}[n..\infty]$ for some $w \in \Sigma^*$, $n \geq 1$. By Lemma 4, if the words u and \overleftarrow{v} are T-extendable, there is a word w such that uwv is cube-free. We analyze T-extendability in Lemmas 5–7.

► **Lemma 5.** *If a uniform cube-free word u has a right context of length 3, or is right aligned and has a right context of length 2, then u is T-extendable.*

Proof. We assume $|u| \geq 5$; otherwise, u is a factor of \mathbf{T} and there is nothing to prove. Consider two cases.

Case 1: $u = c\theta(v) = cv_1 \dots v_n$, $v_i \in \{ab, ba\}$ for $i \in [1..n]$. W.l.o.g., $v_n = ab$.

Claim 1. At least one of the words ua , $ubab$ is cube free.

Assume that ua is not cube free and thus has a suffix w^3 of period p . Then u has the suffix u' of length $3p - 1$ and period p . If $u' \neq u$, by the cube-freeness of u one has $u[|u|-3p+1] \neq u[|u|-2p+1]$, $u[|u|-3p+2] = u[|u|-2p+2]$. Note that $u[|u|-2p+1..|u|-2p+2] = v_{n-p+1} \in \{ab, ba\}$. Then $u[|u|-3p+1] = u[|u|-3p+2]$, so these two letters do not form a block v_i ; then p must be odd. Consider the suffix $xx = v_{n-p+1} \cdots v_n$ of length $2p$ of u' . Both prefix and suffix of x of length $p - 1$ are concatenations of blocks ab, ba . Hence x consists of alternating letters. Then $x[1] = x[p]$. On the other hand, $v_{n-\lfloor p/2 \rfloor} = x[p]x[1]$. This contradiction proves that $u' = u$ and then $|u| = 3p - 1$. Further, since ua and ubb end with cubes, the only length-2 right context of u is ba , so $uba = c\theta(vb)$ is cube-free. If $ubab$ ends with a cube of period p' , we repeat the above argument for uba to obtain $|uba| = 3p' - 1$. Hence $3p - 1 = 3p' - 3$, which is impossible since the periods are integers. So $ubab$ is cube free and Claim 1 holds.

Claim 2. At least one of the words uaa , $ubabb$ is cube free.

Assume that ua is cube free. If uaa has a suffix w^3 of period p , then w ends with aa (recall that u ends with ab). On the other hand, the leftmost w in the suffix w^3 of uaa ends with $v_{n-p+1} \in \{ab, ba\}$. Thus uaa cannot have a cube as a suffix and hence is cube free. The same argument works for $ubabb$ if $ubab$ is cube free. The reference to Claim 1 concludes the proof.

Assuming that uaa is cube free, we show that the word

$$\mathbf{v} = u\mathbf{T}[6..\infty] = cv_1 \cdots v_{n-1} ab aabbaababba \cdots$$

is cube free and then u is \mathbf{T} -extendable. Depending on v_{n-1} , \mathbf{v} has the non-uniform factor $aabaa$ or $aababaa$, and this factor, denoted by x , has a unique occurrence in \mathbf{v} because u and \mathbf{T} are uniform. Note that $ab\mathbf{T}[6..\infty] = \mathbf{T}[4..\infty]$, and both words $a\mathbf{T}[4..\infty]$ and $aab\mathbf{T}[4..\infty]$ are cube free. So if a cube w^3 is a factor of \mathbf{v} , then Claim 2 implies that x is a factor of w^3 . If x is a factor of w^2 , then x occurs in w^3 at least twice, which is not the case. So $x = w'ww''$, where w' and w'' are nonempty suffix and nonempty prefix of w respectively. Then $w = aba$ if $x = aabaa$, and $w = ababa$ if $x = aababaa$. In both cases a direct check shows that w^3 is not a factor of \mathbf{v} . So we proved that \mathbf{v} is cube free. Assuming that $ubabb$ is cube free, we use the same argument for another suffix of \mathbf{T} :

$$\mathbf{v} = u\mathbf{T}[20..\infty] = cv_1 \cdots v_{n-1} ab babbaabbaba \cdots$$

(here \mathbf{v} contains a unique occurrence of $bbabb$). Note that if $v_n = ba$, then we can take $\mathbf{T}[20..\infty]$ (resp., $\mathbf{T}[6..\infty]$) as the extension of u if ubb (resp., $uabaa$) is cube free.

Case 2. $u = c\theta(v)a = cv_1 \cdots v_n a, v_i \in \{ab, ba\}$ for $i \in [1..n]$.

Case 2.1: $v_n = ab$. The word $u\mathbf{T}[7..\infty] = cv_1 \cdots v_n \mathbf{T}[6..\infty]$ is cube free as in Case 1.

Case 2.2: $v_n = ba$. Since ua ends with a^3 , ub is cube free, right aligned and has a right context of length 2. Then ub is \mathbf{T} -extendable by Case 1, and so is u . \blacktriangleleft

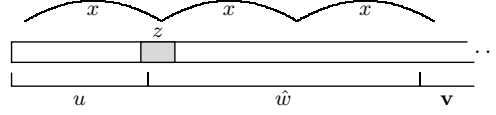
► **Lemma 6.** *If a cube-free word u has a uniform right context w such that $|w| \geq 2|u| + 3$ and w has no prefix $ababa$ or $babab$, then u is \mathbf{T} -extendable.*

Proof. Let \hat{w} be the right aligned prefix of w of length $2|u|$ or $2|u|+1$. We will prove that $u\hat{w}$ is \mathbf{T} -extendable, which implies the result immediately. Suppose $u\hat{w}$ is non-uniform (otherwise, it is \mathbf{T} -extendable by Lemma 5). Then it contains markers, and all of them begin in u by the conditions on w . Let z be the rightmost marker in $u\hat{w}$. W.l.o.g. the first letter of z is a and we can write $u = u'au''$, where $au''w$ begins with this distinguished occurrence of z . The pair $(u'a, u''w)$ satisfies all conditions of the lemma, so for the rest of the proof we

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rename u/a as u and $u''\hat{w}$ as \hat{w} . This renaming retains the value of the word $u\hat{w}$ we analyze; still, \hat{w} is right aligned and $|\hat{w}| \geq 2|u|$. Since \hat{w} has a right context of length at least 2, it is T-extendable by Lemma 5, and, moreover, there is a suffix \mathbf{v} of \mathbf{T} such that $\hat{w}\mathbf{v}$ is cube-free. If \hat{w} is a factor of \mathbf{T} , we choose \mathbf{v} such that $\hat{w}\mathbf{v}$ is a suffix of \mathbf{T} ; otherwise, \mathbf{v} is chosen as in the proof of Lemma 5, Case 1.

Assume to the contrary that $u\hat{w}\mathbf{v}$ contains a cube x^3 ; it starts in u and ends in \mathbf{v} , thus containing the distinguished occurrence of z :



If z occurs in x^3 only once, we have one of two cases, up to symmetry:

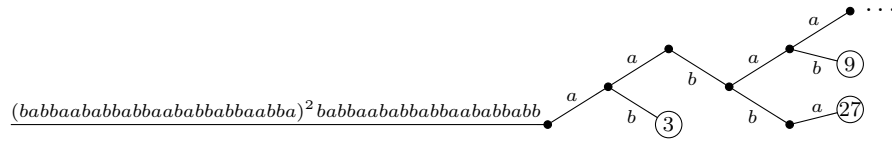
$$\underbrace{\dots aba}_{u} \underbrace{aba}_{\hat{w}\mathbf{v}} \dots \quad \text{or} \quad \underbrace{\dots a}_{u} \underbrace{bab}_{\hat{w}\mathbf{v}} \dots$$

In the first case the condition $|\hat{w}| \geq 2|u|$ is violated; the second case contradicts the choice of \mathbf{v} (here \hat{w} is a factor of \mathbf{T}). Therefore, z must have two occurrences in x^3 at distance $|x|$. If z occurs in x^3 to the left of the distinguished occurrence, then $|x| < |u|$ and x^3 cannot end in \mathbf{v} . Otherwise, x^3 contains exactly two occurrences of z : the distinguished one and another one on the border of \hat{w} and \mathbf{v} . Then x does not contain z , implying $|x| \leq |u| + 3$. On the other hand, $|x| \geq |\hat{w}| - 3$ as the distance between the occurrences of z . Hence $|u| \leq 6$ and $|x| \leq 9$. This leaves, up to symmetry, the following options for x^3 :

$$\begin{array}{ll} |x| = 5 : & abaa \mathbf{baba} ab \dots \\ |x| = 7 : & abba \mathbf{abaabba} abba \dots \end{array} \quad \begin{array}{ll} |x| = 9 : & aabba \mathbf{babaabba} abba \dots \\ & ababba \mathbf{abaabba} abba \dots \end{array}$$

The factor between the marginal letters of markers contains \hat{w} ; in first three cases, this factor occurs in \mathbf{T} but $\hat{w}\mathbf{v}$ is not a suffix of \mathbf{T} , contradicting the choice of \mathbf{v} . In the last case \hat{w} is not a factor of \mathbf{T} , so \mathbf{v} is chosen as in the proof of Lemma 5; hence the marker on the border of \hat{w} and \mathbf{v} must be followed by bb , not ba . This contradiction finishes the proof. ◀

Some right-extendable words have no long uniform right contexts, as Fig. 1 shows. However, a weaker property is enough for our purposes.



■ **Figure 1** A right-extendable word of length 76 having no long uniform right contexts: all its infinite right contexts begin with the marker $aabaa$.

► **Lemma 7.** *Every cube-free word having an infinite right context with finitely many markers is T-extendable.*

Proof. Let u be the word and \mathbf{w} be its context from the conditions of the lemma. The finiteness of the number of markers allows us to write $\mathbf{w} = w_1\mathbf{v}$, where \mathbf{v} is uniform. Then uw_1 has an infinite uniform right context, and hence is T-extendable by Lemma 6. Then u is T-extendable as well. ◀

Thus if Theorem 3 holds, then Lemmas 7 and 4 imply Theorem 1.

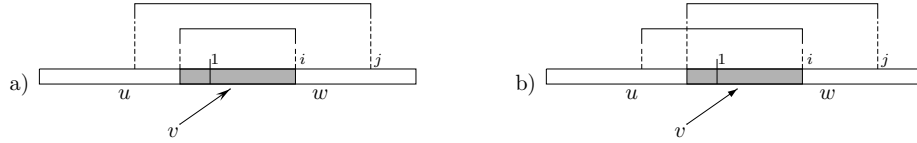
3.2 Proof of Theorem 3

We prove Theorem 3 by reductio ad absurdum; to obtain a contradiction, we use the following lemma on cube-free words over an arbitrary alphabet. (For Theorem 3, bounding k in Lemma 8 by *any* function of n would be sufficient; however, better bounds can be useful for the algorithmic applications.)

► **Lemma 8.** *Let u be a cube-free word of length n over an arbitrary fixed alphabet and let u have a length- k right context w with the following property: for each $i = 1, \dots, k$, there exists an integer $p_i \geq 2$ such that the suffix of length $3p_i - 2$ of the word $u \cdot w[1..i]$ has period p_i and, moreover, $p_i \neq p_{i+1}$. Then $k = O(\log n)$; more precisely, $k \leq \max\{1, 8.13 \log n - 15.64\}$.*

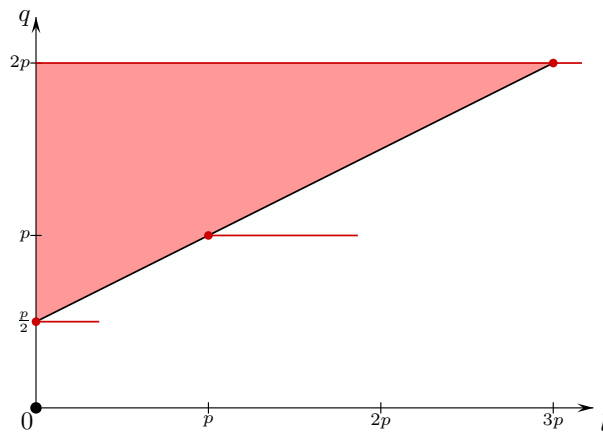
Proof. In the proof we can assume $k \geq 2$. Let $1 \leq i < j \leq k$, $p = p_i$, $q = p_j$, $l = j - i$, and let v be the intersection of the periodic suffixes of $u \cdot w[1..i]$ and $u \cdot w[1..j]$ (see Fig. 2a,b). If $|v| \geq p + q - \gcd(p, q)$, then v has the period $\gcd(p, q)$ by the Fine–Wilf property (Lemma 2). If $p \neq q$, this means that the root of the longer periodic suffix is an integer power of a shorter word; thus uw contains a cube, which is impossible. If $p = q$, then we are in the situation shown in Fig. 2b, and the union of two suffixes has period p and the length $3p - 2 + l$. In this case, $l \geq 2$ by conditions of the lemma, so we again obtain a cube. Thus we conclude that

$$|v| \leq p + q - \gcd(p, q) - 1. \quad (1)$$



■ **Figure 2** The mutual location of periodic factors in the word uw (Lemma 8).

The case in Fig. 2b corresponds to $|v| = 3q - 2 - l$. Comparing this condition to (1), we get $q \leq \frac{p+l}{2}$ (but $q = p$ only if $l \geq 2p - 1$ and $q = p/2$ only if $l \geq p/2 - 1$). Similarly, the case in Fig. 2a corresponds to $|v| = 3p - 2$ and we get $q > 2p$ from (1). Thus, all possible values of q are outside the red area in Fig. 3.



■ **Figure 3** The restrictions on periods of periodic factors in the word uw (Lemma 8).

Now we estimate how many elements of the sequence $\{p_1, \dots, p_k\}$ can belong to the range $[p..2p]$ for some fixed $p \geq 2$. This is an analog of [13, Lemmas 4,5] and [14, Lemma 9]. Let

$i_0 < i_1 < \dots < i_s$ be the list of all positions such that the periodic suffix of $u \cdot w[1..i_j]$ has the period from the range $[p..2p]$; let q_0, \dots, q_s denote these periods. Then Fig. 3 gives us the lower bound for the distance $l_j = i_{j+1} - i_j$ between consecutive positions from the list:

$$l_j \geq 2q_{j+1} - q_j; \quad l_j \geq 2q_{j+1} - 1 \text{ if } q_{j+1} = q_j; \quad l_j \geq q_{j+1} - 1 \text{ if } q_{j+1} = q_j/2. \quad (2)$$

The densest packing of the numbers i_j , satisfying the restrictions (2), is achieved for $q_0 = 2p-1$, $q_1 = q_3 = q_5 = \dots = p$, $q_2 = q_4 = q_6 = \dots = p+1$: one can take $i_0 = 1$, $i_1 = 2$, and $i_{2j} = i_{2j-1} + p + 2$, $i_{2j+1} = i_{2j} + p - 1$ for all subsequent positions. Since $i_s \leq k$, we have $\lceil \frac{s-1}{2} \rceil \cdot (p+2) + \lfloor \frac{s-1}{2} \rfloor \cdot (p-1) + 2 \leq k$, implying the upper bound for the number of periods from the range $[p..2p]$:

$$s+1 \leq \frac{2(k-2)}{2p+1} + 2. \quad (3)$$

Since $3p-2 \leq |uw| = n+k$, the maximum possible value of p is $\lfloor \frac{n+k+2}{3} \rfloor$. We partition all possible periods into r ranges of the form $[p..2p]$:

$$[2..4], [5..10], [11..22], \dots, [3 \cdot 2^{r-1} - 1.. \lfloor \frac{n+k+2}{3} \rfloor].$$

The number of ranges thus satisfies $r \leq \log \frac{n+k+2}{9} + 1$. The sum of the upper bounds (3) for all ranges is at least k ; observing that the number $2p+1$ in (3) is the first period from the range next to $[p..2p]$, we can write

$$k \leq 2(k-2) \cdot \sum_{i=1}^r \frac{1}{3 \cdot 2^{i-1}} + 2r. \quad (4)$$

The sum in (4) is bounded by $\frac{1}{5} + \frac{1}{11} + \frac{1}{23} + \frac{1}{47} \cdot \sum_{i=0}^{\infty} \frac{1}{2^i} < 0.377$; substituting this value and the upper bound for r , we get

$$0.246(k-2) \leq 2 \log \frac{n+k+2}{9}. \quad (5)$$

For $k \geq n-1$, (5) implies $0.246(k-2) \leq 2 \log \frac{2k+3}{9}$, but this inequality fails for $k \geq 2$. So $k \leq n-2$ and we replace (5) with the inequality $0.246(k-2) \leq 2 \log \frac{2n}{9}$; it gives, after arithmetic transformations, the required bound on k . \blacktriangleleft

Proof of Theorem 3. For the sake of contradiction, assume that all infinite right contexts of some right-extendable cube-free word u contain infinitely many markers. W.l.o.g. we can assume that u ends with a marker (if not, choose a prefix v of an infinite right context of u such that uv ends with a marker, and replace u with the word uv having the same property of right contexts). Let z be the marker which is a suffix of u . For example, if u is the word written in the “trunk” of the tree in Fig. 1, then $z = bbabb$.

By our assumption, u has no infinite uniform right contexts. Thus u has finitely many uniform right contexts (in Fig. 1 such contexts are λ , a , aa , aab , $aaba$, and $aabb$). Other uniform words, being appended to u , produce cubes; in Fig. 1, appending a word beginning with ab (resp., $aabab$, $aabba$) gives the cube $(bab)^3$ (resp., $(babbaabab)^3$, $(babbaababbabbaababbabbaabba)^3$). All these cubes contain markers and illustrate three types of cubes with respect to the occurrences of markers (x below denotes the root of the cube):

- *mini*: x^2 contains no markers, x^3 contains a marker (example: $x = bab$);
- *midi*: x contains no markers, x^2 contains a marker (example: $x = babbaabab$);
- *maxi*: x contains markers (example: $x = babbaababbabbaababbabbaabba$);

Note that mini cubes are exactly those having the root $x \in \{ab, ba, aba, bab\}$. Further, the intersection of two markers in a cube-free word is either empty or one-letter; this fact implies that each midi cube contains exactly two markers.

We call w a *semi-context* of u if uw ends with a cube but $u \cdot w[1..|w|-1]$ is cube free. Next we show the following fact:

- (*) The word u has two distinct uniform semi-contexts w_1 and w_2 such that uw_1 and uw_2 end with midi or maxi cubes.

To prove (*) we need a case analysis. W.l.o.g., z begins with a . Let \mathbf{w} be an infinite right context of u .

Case 1: $z = aabaa$. We have $u = \cdots baabaa$, so \mathbf{w} cannot begin with a or baa because $u\mathbf{w}$ is cube-free. So \mathbf{w} begins with bba , $babba$, or $baba$. In the first case, some prefixes of words

$$u\mathbf{T}[2..\infty] = \cdots baabaa bba baababba \cdots, \quad (6)$$

$$u\mathbf{T}[22..\infty] = \cdots baabaa bba abbabaab \cdots \quad (7)$$

must end with cubes. The longest common prefix $ubba$ of these words is cube free as a prefix of $u\mathbf{w}$, so these cubes are different, contain the marker z , and are not mini. Hence we can take some prefixes of $\mathbf{T}[2..\infty]$ and $\mathbf{T}[22..\infty]$ as the semi-contexts required in (*). If \mathbf{w} begins with $babba$, the same result is obtained with prefixes of the words

$$u\mathbf{T}[12..\infty] = \cdots baabaa babba baabbaab \cdots, \quad (8)$$

$$u\mathbf{T}[20..\infty] = \cdots baabaa babba abbabaab \cdots \quad (9)$$

Finally, if \mathbf{w} begins with $baba$, we can take one word from each pair (say, $u\mathbf{T}[2..\infty]$ and $u\mathbf{T}[12..\infty]$). Their longest common prefix is the cube-free word ub , so the cubes given by the corresponding semi-contexts are distinct.

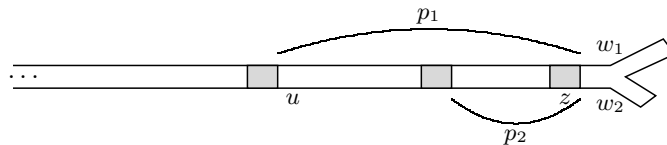
Case 2: $z = ababa$. Here $\mathbf{w} = ab \cdots$. Taking the pair of words

$$u\mathbf{T}[7..\infty] = \cdots ababa ab baababba \cdots, \quad (10)$$

$$u\mathbf{T}[19..\infty] = \cdots ababa ab abbaabba \cdots, \quad (11)$$

we achieve the same result as in Case 1: some prefixes of these words end with midi or maxi cubes, and these cubes are distinct because the common prefix uab of the presented words is cube-free. Thus, (*) is proved.

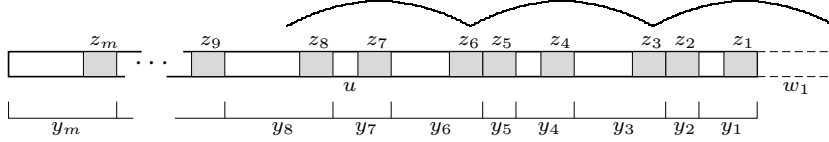
Now take the semi-contexts w_1, w_2 given by (*) such that uw_1 and uw_2 end with cubes x_1^3 of period p_1 and x_2^3 of period p_2 respectively. Let w be the longest common prefix of w_1 and w_2 ; w.l.o.g., $w_1 = waw'_1$, $w_2 = wbw'_2$. In both x_1^3 and x_2^3 , the suffix z of u is the rightmost marker and hence matches an earlier occurrence of the same marker in u . These occurrences are different, because z is followed by wa in x_1^3 and by wb in x_2^3 . In particular, $p_1 \neq p_2$. W.l.o.g., $p_1 > p_2$; then x_1 contains z and so x_1^3 is maxi (see Fig. 4).



■ **Figure 4** Semi-contexts w_1 and w_2 of the word u : periods of cubes and corresponding markers. Three grey factors are markers equal to z , other markers are not shown.

XX:10 Transition Property For Cube-Free Words

Let $z_1 = z, z_2, \dots, z_m$ be all markers in u , right to left. We factorize u as $u = y_m \cdots y_2 y_1$, where y_i begins with the first letter after z_{i+1} (y_m is a prefix of u) and ends with the last letter of z_i (even if z_{i+1} and z_i overlap); see Fig. 5 for the example. Assume that z_{j+1} matches z_1 in the maxi cube x_1^3 (note that $j \geq 2$, because a marker with a smaller number matches z_1 in x_2^3 ; $j = 3$ in Fig. 5). Then z_{j-1}, \dots, z_1 are in the rightmost occurrence of x_1 , and z_j is either also in this occurrence or on the border between the middle and the rightmost occurrences (in Fig. 5, the latter case is shown). Depending on this, x_1^3 contains either $3j$ or $3j-1$ markers. Further, we see that w_1 is a prefix of y_j , $y_1 = y_{j+1}, \dots, y_{2j-2} = y_{3j-2}$.



■ **Figure 5** Marker-based factorization of the word u . Markers are grey, arcs indicate the cube after appending w_1 to u .

Let us extend u to the right by a context y_0 such that uy_0 is right extendable, y_0 ends with a marker z_0 , and all proper prefixes of y_0 are uniform. Applying all the above argument to uy_0 and its factorization $y_m \cdots y_1 y_0$, we get another maxi cube (say, x_0^3) and the corresponding set of equalities between y_i 's. Note that $y_0 \neq y_j$: as was mentioned in the previous paragraph, y_j has the prefix w_1 , while y_0 cannot have this prefix because uw_1 contains a cube.

Let us iterate the procedure of appending a context k times, getting a right-extendable word $uy = y_m \cdots y_1 y_0 \cdots y_{1-k}$ as the result (according to our assumption on u , the number k can be arbitrarily big). Now consider the finite alphabet $\Gamma = \{y_m, \dots, y_1, y_0, \dots, y_{1-k}\}$ and let $U = y_m \cdots y_2 y_1$, $Y = y_0 \cdots y_{1-k}$ be words over Γ . They are cube free and Y is a length- k right context of U . Each word $U \cdot Y[1..i]$ ends with a suffix having some period p_i and length $3p_i - 2$ or $3p_i - 1$. In addition, $p_i \neq p_{i+1}$, because $y_{-i} \neq y_{p_i-i}$. So all conditions of Lemma 8 are satisfied, and we apply it to get an upper bound on k . The existence of this bound contradicts our assumption that all infinite right contexts of u have infinitely many markers. The theorem is proved. ◀

4 Transition Property for Big Alphabets

Here we extend the results of the previous section to arbitrary finite alphabets.

► **Theorem 9.** *For every $d \geq 3$ and every pair (u, v) of d -ary cube-free words such that u is right extendable and v is left extendable, there exists a d -ary word w such that uwv is cube free.*

As in the binary case, we use an auxiliary theorem about the existence of a context with finitely many markers (but the markers are different now).

► **Theorem 10.** *Let $d \geq 3$. Every right-extendable cube-free word $u \in \Sigma_d^*$ has an infinite right context with finitely many occurrences of all letters except for a and b .*

Proof. We follow the main idea of the proof of Theorem 3 and use the same notation. The difference, which actually simplifies the argument, is that the role of markers is now played by the c -letters c_1, \dots, c_{d-2} . Aiming at a contradiction, assume that all infinite right contexts

of some right-extendable cube-free word u contain infinitely many c -letters. W.l.o.g. we can assume that u ends with a c -letter; we denote this letter by z . As in the proof of Theorem 3, we let $z_1 = z, z_2, \dots, z_m$ be all markers in u , right to left, and factorize u as $u = y_m \cdots y_2 y_1$, where y_i begins with the first letter after z_{i+1} and ends with the last letter of z_i .

By our assumption, u contains finitely many contexts from $\{a, b\}^*$; then it has two semi-contexts $w_1 = waw'_1, w_2 = bw'_2 \in \{a, b\}^+$ (each of the words w, w_1, w_2 may be empty). Let x_1^3, x_2^3 be suffixes of uw_1 and uw_2 respectively, and w.l.o.g. $|x_1| = p_1 > p_2 = |x_2|$. The suffix zw_1 of x_1 matches some earlier occurrence of zw_1 in u ; same for the suffix zw_2 of x_2 . As in the proof of Theorem 3 we see that $z_{j+1} = z_1$ for some $j \geq 2$, w_1 is a prefix of y_j , and the equalities $y_1 = y_{j+1}, \dots, y_{2j-1} = y_{3j-1}$ hold.

Next we extend u to the right by a context y_0 such that uy_0 is right extendable, y_0 ends with a marker z_0 , and all proper prefixes of y_0 are over $\{a, b\}$. Applying all the above argument to uy_0 and its factorization $y_m \cdots y_1 y_0$, we get another cube x_0^3 and the corresponding set of equalities between y_i 's. Again, $y_0 \neq y_j$, since y_j has the prefix w_1 , while y_0 has not. After iterating the procedure of appending a context k times, we obtain a right-extendable word $uy = y_m \cdots y_1 y_0 \cdots y_{1-k}$ and consider the words $U = y_m \cdots y_2 y_1$, $Y = y_0 \cdots y_{1-k}$ over the alphabet $\Gamma = \{y_m, \dots, y_1, y_0, \dots, y_{1-k}\}$. They are cube free and Y is a length- k right context of U . Each word $UY[1..i]$ ends with a suffix having some period p_i and length $3p_i - 1$. In addition, $p_i \neq p_{i+1}$, because $y_{-i} \neq y_{p_i-i}$. So we can apply Lemma 8 to get an upper bound on k . The existence of this bound contradicts our assumption that all infinite right contexts of u have infinitely many c -letters. Hence u has an infinite context with finitely many c -letters, as required. ◀

Proof of Theorem 9. By Theorem 10, the word $u \in \Sigma_d^*$ has an infinite right context with finitely many c -letters. First we note that we can choose such a context containing a c -letter (if a context \mathbf{w} is over Σ_2 , one can get another context of u replacing, say, the letter $\mathbf{w}[|u|]$ with c_1). So we can write this context \mathbf{w} as $x\mathbf{u}_1$, where x ends with a c -letter and $\mathbf{u}_1 \in \Sigma_2^\infty$. Let u_1 be the prefix of \mathbf{u}_1 of length $\lceil |ux|/2 \rceil$. In the same way, we take a right context $\overleftarrow{y}\overleftarrow{v}_1$ of \overleftarrow{v} and the prefix $\overleftarrow{v}_1 \in \{a, b\}^*$ of \overleftarrow{v}_1 of length $\lceil |yv|/2 \rceil$. Then the binary words u_1, v_1 are cube free, u_1 is right extendable, and v_1 is left extendable. Applying Theorem 1, we take a binary transition word w_1 such that $u_1 w_1 v_1$ is cube free. Then $s = xu_1 w_1 v_1 yv$ is cube free. Indeed, x ends with a c -letter, y begins with a c -letter, and these c -letters are separated by a cube-free word over Σ_2 . Hence a cube in s , if any, must contain one of these c -letters. But the lower bounds on $|u_1|$ and $|v_1|$ imply that this c -letter cannot match another c -letter to produce a cube (recall that uxu_1 and $v_1 yv$ are cube free). Thus s is cube-free and we obtain a transition word $xu_1 w_1 v_1 y$ for the pair (u, v) . ◀

5 Solving the Restivo-Salemi Problems and Future Work

To give the solutions to the Restivo–Salemi Problems 2, 4, and 5, recall the solution to Problem 1a [4]: a d -ary α -power-free word u is right extendable iff it has a right context of length $f_{\alpha,d}(|u|)$ for some computable function $f_{\alpha,d}$. Algorithm 1 below solves Problem 4.

The natural next step is to find an *efficient* algorithm for Problem 4. The function $f_{3,d}(n)$ is sublinear, but the search space is still of size $2^{n^{\Omega(1)}}$. The possible way to a polynomial-time solution is to strengthen the connection with Lemma 8 to show that it is sufficient to process the contexts of length $O(\log n)$, where $n = \max\{|u|, |v|\}$.

For Problem 2, the first step is the reduction to the binary case. Let $u \in \Sigma_d$, $d \geq 3$, be a right-extendable cube-free word; we write $u = u'cu''$, where c is the rightmost c -letter in u . We check all cube-free words $w \in \Sigma_2^*$ such that $u''w$ is right extendable and $|u''w| = \lceil |u|/2 \rceil$

Algorithm 1 : Deciding the existence of a transition word for cube-free words $u, v \in \Sigma_d$

- For all words $w \in \Sigma_d$ such that $|w| \leq f_{3,d}(|u|)$ check whether w is a right context of u
 - If a context w with the suffix v is found, return “yes”
 - If no context of length $f_{3,d}(|u|)$ is found, return “no”
 - Else $\triangleright u$ is right extendable
 - For all words $w \in \Sigma_d$ such that $|w| \leq f_{3,d}(|v|)$ check whether w is a left context of v
 - If a context w with the prefix u is found, return “yes”
 - If no context of length $f_{3,d}(|v|)$ is found, return “no”
 - Else return “yes” $\triangleright v$ is left extendable; apply Theorem 1 or 9
-

for being right contexts of u . If w is a right context of u , then any (binary) right context of $u''w$ is a right context of uw , so the problem is reduced to binary words. If no word w suits, we take the shortest right context of u of the form vc , where $v \in \Sigma_2^*$, $c \in \{c_1, \dots, c_{d-2}\}$ such that the word $u_1 = uvc$ is right extendable; such a context can be found in finite time because $|v| < |u|/2$. Then we replace u by u_1 and repeat the search of long binary right contexts. By Theorem 10, we will succeed after a finite number of iterations, and Lemma 8 gives the upper bound on the maximum number k of iterations depending on $|u|$. Thus we end this step getting a word $y\hat{u}$ such that $uy\hat{u}$ is cube free, $\hat{u} \in \Sigma_2^*$, and all binary right contexts of \hat{u} are right contexts of $uy\hat{u}$. If u is binary, we skip this step setting $\hat{u} = u$.

On the second step we further reduce the problem to uniform words. We act as in the first step, using Theorem 3 and Lemma 6. Namely, we check for uniform contexts and if \hat{u} has no uniform context w of length $2|\hat{u}| + 3$ such that $\hat{u}w$ is right extendable and w has no prefix *ababa*/*babab*, we append the shortest context v ending with a marker, repeating the search for $\hat{u}_1 = \hat{u}v$. Theorem 3 guarantees that we will find the required uniform context in at most k iterations, where k is as in Lemma 8. Thus at this step we build a right context $\hat{y}\hat{w}$ of \hat{u} such that $\hat{u}\hat{y}\hat{w}$ is right extendable, $|\hat{w}| \geq 2|\hat{u}\hat{y}| + 3$ and \hat{w} is uniform and has no prefix *ababa*/*babab*.

Finally we choose, as described in Lemma 6, a suffix $\mathbf{T}[r..\infty]$ of \mathbf{T} which is a right context of \hat{w} : if $\hat{w} = \mathbf{T}[i..j]$ for some i, j , then we take $r = j + 1$, otherwise the choice is performed according to Case 1 in the proof of Lemma 5. Now Lemma 6 guarantees that $\hat{u}\hat{y}\hat{w}\mathbf{T}[r..\infty]$ is cube free. Thus the infinite right context of the original word u is given by the finite word $Y = y\hat{u}\hat{y}\hat{w}$ and the number r . The above description is summarized below as Algorithm 2.

Again, the natural direction of the future work is to make Algorithm 2 efficient.

Finally we approach Problem 5. We first run Algorithm 1, which can provide us with an example of a transition word if u or \overleftarrow{v} is not right extendable. If both u, \overleftarrow{v} are right extendable, we run for each of them Algorithm 2, getting Y_1, Y_2, r_1, r_2 such that $uY_1\mathbf{T}[r_1..\infty]$ and $\overleftarrow{\mathbf{T}}[\infty..r_2]Y_2v$ are cube free. It remains to use Lemma 4: take big enough r'_1, r'_2 and find a word w such that $\mathbf{T}[r_1..r'_1]w\overleftarrow{\mathbf{T}}[r'_2..r_2]$ is a factor of \mathbf{T} and a transition word for the pair (uY_1, Y_2v) ; the uniform recurrence of \mathbf{T} ensures that the word w can be found in finite time. Thus $Y_1\mathbf{T}[r_1..r'_1]w\overleftarrow{\mathbf{T}}[r'_2..r_2]Y_2$ is the transition word for the pair (u, v) , so Problem 5 is solved.

Once again, it is clear that some steps of the above solution can be significantly sped up, so it would be nice to finally get a polynomial-time algorithm for Problem 5 (and thus for Problems 1, 2, 4 as well). From the experimental study we learned that if a length- n cube-free word is not right extendable, then likely not only all its right contexts have the length $O(\log n)$, but the number of such contexts is $O(\log n)$. The proof of this fact would lead to a linear-time solution of Problem 1.

Algorithm 2 : Finding an infinite right context of a right-extendable cube-free word $u \in \Sigma_d$

- $U \leftarrow u, Y \leftarrow \lambda$
 - If U is not binary ▷ first step
 - While U has no long binary right context
 - * Find the shortest right context vc ending with a c -letter
 - * $U \leftarrow Uvc, Y \leftarrow Yvc$
 - $\hat{u} \leftarrow$ long binary right context of $U, Y \leftarrow Y\hat{u}$
 - Else $\hat{u} = u$
 - While \hat{u} has no long uniform right context without prefix $ababa/babab$ ▷ second step
 - Find the shortest right context v such that $\hat{u}v$ ends with a marker
 - $\hat{u} \leftarrow \hat{u}v, Y \leftarrow Yv$
 - $\hat{w} \leftarrow$ long uniform right context of \hat{u} without prefix $ababa/babab, Y \leftarrow Y\hat{w}$
 - Find r such that $\mathbf{T}[r..\infty]$ is a right context of \hat{w} ▷ final step
 - return Y, r
-

Another obvious continuation of the current research is the study of the same problems for other power-free languages. One line is to use Thue-Morse words to solve Problems 2, 4, and 5 for other binary power-free languages. For example, we are able to extend the results of Section 3.1 to α -power-free binary words for any $\alpha \in (5/2, 3]$, changing only some constants. Another line is to obtain similar results for ternary square-free words, in the absence of such a strong tool as Thue-Morse words.

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